XII.

NOTE ON THE DETERMINATION OF THE LAW OF PROPAGATION OF HEAT IN THE INTERIOR OF A SOLID BODY.

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Newton's experiments upon the amount of heat communicated from a body A to a neighboring body B, at a lower temperature than A, led him to think that this amount was directly proportional to the difference of temperature between the two bodies. In 1805, Biot, assuming that Newton's results were reliable, conceived that the same law must hold for the communication of heat between two neighboring molecules in the interior of a solid body, and he compared the observed temperatures at different points of a long bar heated at one end with the temperatures calculated on the assumption that the flux of heat in the direction x is represented by

$$-\varkappa \frac{dv}{dx}$$

where $\varkappa$ is constant for the same body and v is the temperature of the point under consideration. Fourier — whose "Théorie de la Chaleur" was written in 1811, but not published until 1822 — followed Biot in assuming

$$-\varkappa \frac{dv}{dx}$$

to represent the flux of heat in the inside of a body, and

$$-h \frac{dv}{dx}$$

the radiation at its surface, where $\varkappa$ and h are different constants which he calls respectively the "conducibilité propre" and "conducibilité relative à l'air atmosphérique." Just before Fourier's work was published, MM. Dulong and Petit showed that the amount of heat communicated from one body to another depends not only upon the

difference of their temperatures, but also upon the absolute temperatures of each. Poisson published, in 1835, his "Théorie de la Chaleur," in which he assumed that the expression which Dulong and Petit had given for the loss of heat from radiation also represents the passage of the heat from molecule to molecule in the interior of the body. Libri, shortly before Poisson's book was published, presented to the Academy of Sciences a paper in which he assumed that Fourier and Biot were correct in their hypothesis that the internal flux of heat could be written

$$-\varkappa \frac{dv}{dx},$$

but that the law of extra radiation was that stated by Dulong and Petit. In 1837, Kelland published his "Theory of Heat." He applied Libri's hypotheses to the problem of determining the final distribution of heat in a ring, and showed that the solution thus arrived at was not very different from that which Fourier had determined. In other respects, Kelland simply gave Fourier's work with corrections, as his object was to furnish a book for students. In 1841, Professor Kelland made a report to the British Association for the Advancement of Science "On the Present State of our Theoretical and Experimental Knowledge of the Laws of Conduction of Heat."

In this report, Kelland says that, although objections might be made to the particular assumptions of Fourier, Libri, and Poisson, it is very probable that the flux of heat in the interior of a body may be written

$$-c \frac{df(v)}{dx},$$

where c is a constant depending upon the body and $f(v)$ is some undetermined function of the temperature. Kelland assumed a particular value for $f(v)$, and compared the temperatures calculated from the different hypotheses of Fourier, Libri, Poisson, and himself, with the corresponding temperatures observed by Biot in his experiments upon long bars. This comparison does not give the preference to any one of the different assumptions. Since 1841, nothing of any importance has been done, so far as I know, in the general theory of heat conduction. Lamè, whose "Théorie de la Chaleur" was published in 1862, follows Fourier in his hypotheses, and those writers who, like Sir William Thomson, have had occasion to discuss practical questions about the cooling of bodies, have also made the same assumptions,

since the error thus introduced into their calculations is necessarily less than those arising from errors in observing the phenomena.

Dulong and Petit's experiments showed that Fourier's assumption with regard to the flux of heat at the surface of a body due to radiation was wrong, and Principal Forbes's experiments upon heated metallic bars showed that, in order to write the flux of heat in the interior of a body

$$-\varkappa \frac{dv}{dx},$$

$\varkappa$ must be regarded as a function of the temperature. Forbes's experiments evidently offer no objection to Kelland's hypothesis, for

$$-\varphi(v)\frac{dv}{dx} \quad \text{and} \quad -c\frac{df(v)}{dx}$$

are equivalent expressions, if

$$\varphi(v) = cf'(v).$$

The first step in determining the form of the function f is made by showing that it must satisfy a differential equation which when the heated body is at its final state, reduces to Laplace's Equation.

Consider the element of volume $dxdydz$, which has one of its angles at the point (x, y, z) and its diagonally opposite angle at $(x + dx, y + dy, z + dz)$. During the instant dt, the flux of heat across that face of the element which contains the point (x, y, z) and is parallel to the coördinate plane xy, is

$$F(v, z) = -c\frac{df(v)}{dz}dxdydt.$$

The amount of heat which flows out at the opposite face of the element is obtained by developing $F(v, z)$ by Taylor's Theorem:

$$F(v + dv, z + dz) = -c\frac{df(v)}{dz}dxdydt - c\frac{d^2f(v)}{dz^2}dxdydzdt.$$

The flux across the second face is less than the flux across the first face by

$$c\frac{d^2f(v)}{dz^2}dxdydzdt.$$

Considering each of the other pairs of opposite faces, it is evident that in the instant dt a quantity of heat equal to

$$cdxdydz\left(\frac{d^2f(v)}{dx^2} + \frac{d^2f(v)}{dy^2} + \frac{d^2f(v)}{dz^2}\right)dt$$

have been added to the element. Let Q be the total amount of heat in the molecule, then

$$d_t Q = - \, c dx dy dz \left(\frac{d^2 f(v)}{dx^2} + \frac{d^2 f(v)}{dy^2} + \frac{d^2 f(v)}{dz^2} \right) dt$$

If Laplace's Operator is written "$- \nabla^2$,"

$$\frac{d_t Q}{dt} = - \, c \nabla^2 f(v) \, dx dy dz.$$

Let s be the specific heat of the body which Dulong and Petit have shown to be a function of the temperature, and let $s = \Psi(v)$, then

$$d_t Q = d_t v \, . \, \Psi s \, . \, dx dy dz,$$

$$\Psi(v) \, . \, \frac{d_t v}{dt} = - \, c \nabla^2 f(v)$$

If $\qquad r^2 = x^2 + y^2 \qquad$ and $\qquad \varphi = \tan^{-1} \frac{y}{x}$

$$- \nabla^2 f(v) = \frac{d^2 f(v)}{dr^2} + \frac{d^2 f(v)}{r^2 \, . \, d\phi^2} + \frac{d f(v)}{r \, . \, dr} + \frac{d^2 f v}{dz^2}$$

$$\therefore \, \Psi(v) \, \frac{d_t v}{dt} = + \, c \left(\frac{d^2}{dr^2} + \frac{d}{r \, . \, dr} + \frac{d^2}{r^2 d\phi^2} + \frac{d^2}{dz^2} \right) f(v)$$

If the body has reached its final state, the element loses as much heat in any given time as it gains, so that $f(v)$ must satisfy Laplace's Equation, or

$$\nabla^2 f(v) = 0.$$

Consider a thin plate of metal of practically infinite extent, and of which all points are at a uniform temperature. Let this plate be laid upon and covered with some perfectly non-conducting material, so that there can be no flux of heat perpendicular to the plane of the plate, and let a single point be heated by means of a copper wire pushed through the non-conducting material upon which the plate lies.

There will be a flux of heat from the heated point in all directions in the plane of the plate; and, if the plate is homogeneous, the flux will be the same in all azimuths.

After the plate has reached its final state, the amount of heat added to each element of the plate will be the same that flows out of it, and $dQ = 0$. If the plate lies in the coördinate plane xy, there will be no flux in the direction of the axis of z, and hence

$$\frac{d^2 f(v)}{dz^2} = 0;$$

and, if the plate is homogeneous,

$$\frac{d^2 fv}{d\phi^2} = 0.$$

Therefore $f(v)$ must be a solution of the differential equation

$$\frac{d^2 f(v)}{dr^2} + \frac{1}{r} \cdot \frac{df(v)}{dr} = 0$$

or $$f(v) = A + B \log r,$$

and the flux $$= - c\,\frac{df(v)}{dr} = - \frac{cB}{r}$$

Consider a second plate of metal in every way like the first, only that it is heated at two points by means of a Y shaped piece of copper which is itself heated at its stem. The two arms of the Y are pushed through the non-conducting material and are of equal lengths, so that the two points shall be equally heated.

If r_1 and r_2 are the distances of any point from the two heated points, it is evident from the theory of conjugate functions that

$$f(v) = A' + B' \log r_1 r_2$$

$f(v)$ is constant along any curve of the system $[r_1 r_2 = \text{const.}]$ If a is the distance of the heated points from each other, the equation of the system of curves for any one of which $f(v)$ is constant may be written

$$(x^2 + y^2)\left((x - a)^2 + y^2\right) = k.$$

Before the plate is imbedded in the non-conducting material, let it be covered with a thin layer of a mixture of paraffine, rosin, and wax, and after it has been heated long enough to have sensibly reached its final state let the source of heat be removed; then, if there is a clean line of demarcation between the wax that has been melted and that which has not, the form of one of the isothermals can be studied at leisure. Wherever $f(v)$ is constant, v must be constant unless $f(v)$ is an equation of an infinitely high degree, which is inadmissible; and conversely, if v is constant along any curve, $f(v)$ must also be constant at all points on that curve. If the isothermal traced by the melted wax is a curve whose equation is $r_1 r_2 = c$, it will be safe to assume that the flux of heat in the interior of a body is

$$- c\,\frac{df(v)}{dx},$$

as Kelland assumed. By use of a suitable arrangement, it would be very easy to use such ranges of temperature as should make the experiment decisive. If the points were unequally heated by accident, the wax curve would not be symmetrical with respect to a line perpendicular to that joining the two heated points. There could be no trouble with lack of homogeneity of the plate, as preliminary experiments show that from a single heated point the curve is perfectly symmetrical and distinct, with a probable error in finding the line of demarcation practically insensible. I am indebted to Professor Gibbs for the suggestion that the New Hampshire Infusorial Earth might be advantageously used as an almost perfect non-conductor of heat.

Whatever might be the result of a series of experiments like those referred to above, it is evident, from the differential equation, that Fourier's solution cannot be a correct one. If the uniform temperature of the plate were taken as the zero of the scale, the temperature of any point in the plate due to the two heated points would be the sum of the temperatures that would be given to the same point if each heated point acted alone. On this hypothesis, it would be very easy to find a point in the plate so situated that, when the plate had reached its final state, the point would have a temperature nearly double that which either of the heated points have, which is manifestly absurd. It does not appear that writers upon this subject have noticed this fact.

If experiment shows, as it probably will, that the propagation of heat in the interior of a solid is determined by the expression

$$-c\,\frac{df(v)}{dx},$$

it will not be hard to determine $f(v)$ experimentally. The infusorial earth, it is believed, will prevent any sensible loss of heat from conduction or radiation, so that the flow of heat in the plate will be determined solely by the law of internal flux. Let a plate of any metal other than copper be heated at a single point, and after it has reached its final state let the temperatures of the different points of the plate be determined relatively by means of the thermoelectric currents obtained by touching different points of the plate by the copper terminals of a Thomson's Short-Coil Galvanometer. These terminals are to be held by wooden pincers, and pushed through the light infusorial earth so as to explore the points of any line radiating from the point where the heat is applied.

Let r_0, r_1, r_2, r_3, &c., be a series of points in such a line determined by experiment, so that the galvanometer gives the same deflection when one terminal is at r_{n-1} and the other at r_n as it does when the first terminal is at r_1 and the second at r_2. If v is the temperature of all points upon the circle whose radius is r_0, the temperature of all points at a distance from the heated point equal to r_1 will be $(v - \nabla v)$, and all points at a distance of r_n will have a temperature equal to $(v - n\nabla v)$. The temperatures may be determined as a function of r, $[v = \varphi(r)]$ by obtaining the equation of the curve drawn by plotting the temperature as abscissas and the corresponding values of r as ordinates; and the form of the function f may be mathematically obtained from the equation.

$$f\big(\varphi(r)\big) = A' + B' \log r,$$

A' is the value of $f\big(\varphi(1)\big)$.

Kelland unintentionally says that the assumption of

$$-c\,\frac{df(v)}{dx}$$

as the law of flux will only necessitate the writing of $f(v)$ instead of v in Fourier's formulæ. This statement is evidently only true when these formulæ refer to a body which has attained its final state.

Mr. E. B. Lefavour and myself are engaged upon the experimental work laid out in this paper.

Harvard University, April 4, 1877.